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Approximation of Solutions to History-Valued Neutral Functional Differential Equations

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Abstract—In this paper, we consider a class of abstract neutral functional differential equations in a separable Hilbert space and study the approximation of solutions. An example is also given to illustrate the applications of the abstract results.

Keywords—Neutral functional differential equation, Analytic semigroup, Mild solution, Faedo-Galerkin approximation.

1. INTRODUCTION

Let H be a separable Hilbert space and $0 < \tau$, $T < \infty$. For $0 \le t \le T$, $\mathcal{C}_t := C([-\tau, t]; H)$ be the Banach space of all continuous functions from $[-\tau, t]$ into H endowed with the supremum norm,

$$\|u\|_t := \sup_{-\tau \le s \le t} \|u(s)\|, \qquad u \in \mathcal{C}_t, \tag{1.1}$$

where $\|.\|$ is the norm in H. For $u \in \mathcal{C}_T$ and $0 \le t \le T$, $u_t \in \mathcal{C}_0$ be the function defined by $u_t(\theta) = u(t+\theta)$, for $\theta \in [-\tau, 0]$.

We are interested in the following neutral functional differential equation in H,

$$\begin{aligned} \frac{d}{dt}\left(u(t) + g(t, u(t), u_t)\right) + Au(t) &= f(t, u(t), u_t), \qquad 0 < t \le T < \infty, \\ u(t) &= \chi(t), \qquad t \in [-\tau, 0], \end{aligned} \tag{1.2}$$

where $A : D(A) \subset H \to H$ is a closed, densely defined positive definite, self-adjoint linear operator and the nonlinear functions f and g are defined from $[0,T] \times H \times \mathcal{C}_0$ into H and $\chi \in \mathcal{C}_0$.

For the earlier work on existence, uniqueness, regularity, and stability of various types of solutions of differential equations, functional differential equations, and neutral functional differential equations under different conditions, we refer to [1–6] and reference cited in these papers.

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Also, for the works on existence, uniqueness, we refer to [7–11], and references cited in these papers.

Hernandez and Henriquez [2,3] have established some results concerning the existence, uniqueness, and qualitative properties of the solution operator of the following general partial neutral functional differential equation with infinite delay,

$$\frac{d}{dt}\left(u\left(t\right)-g\left(t,u_t\right)\right)=Au\left(t\right)+f\left(t,u_t\right),\qquad t\geq 0,$$
$$u_0=\varphi\in C_0,$$

where A generates an analytic semigroup on a Banach space B, g, and f are continuous functions from $[0,\infty)\times C_0$ into B and for each $u:(-\infty,b]\to B$, $b>0$, and $t\in[0,b]$, u_t represents, as usual, the mapping defined from $(-\infty,0]$ into B by

$$u_t\left(\theta\right)=u\left(t+\theta\right),\qquad \text{for } \theta\in\left(-\infty,0\right].$$

Adimy, Bouzahir and Ezzinibi [1] have studied the existence and stability of solutions of the following general class of nonlinear partial neutral functional differential equations with infinite delay,

$$\begin{aligned}\frac{d}{dt}\left(u\left(t\right)-g\left(t,u_t\right)\right)&=A\left(u\left(t\right)-g\left(t,u_t\right)\right)+f\left(t,u_t\right),\qquad t\geq 0,\\ u_0&=\varphi\in C_0,\end{aligned}\tag{1.3}$$

where the operator A is the Hille-Yosida operator not necessarily densely defined on the Banach space B. The functions g and f are continuous from $[0,\infty)\times C_0$ into B.

The related results for the approximation of solutions may be found in [12–14].

Initial studies concerning existence, uniqueness, and finite-time blow-up of solutions for the following equation,

$$\begin{aligned}u'\left(t\right)+Au\left(t\right)&=g\left(u\left(t\right)\right),\qquad t\geq 0,\\ u\left(0\right)&=\phi,\end{aligned}\tag{1.4}$$

have been considered by Segal [15], Murakami [16], and Heinz and von Wahl [17]. Bazley [18,19] has considered the following semilinear wave equation,

$$\begin{aligned}u''\left(t\right)+Au\left(t\right)&=g\left(u\left(t\right)\right),\qquad t\geq 0,\\ u\left(0\right)&=\phi,\qquad u'\left(0\right)=\psi,\end{aligned}\tag{1.5}$$

and has established the uniform convergence of approximations of solutions to (1.4) using the existence results of Heinz and von Wahl [17]. Goethel [20] has proved the convergence of approximations of solutions to (1.4) but assumed g to be defined on the whole of H. Based on the ideas of Bazley [18,19], Miletta [21] has proved the convergence of approximations to solutions of (1.4). In the present work, we use the ideas of Miletta [21] and Bahuguna [7,8] to establish the convergence of finite-dimensional approximations of the solutions to (1.2).

2. PRELIMINARIES AND ASSUMPTIONS

We assume in (1.2) that the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A)\subset H$ into H such that $D(A)$ is dense in H, A has the pure point spectrum,

$$0<\lambda_0\leq\lambda_1\leq\lambda_2\leq\cdots,$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i=\lambda_i\phi_i\qquad\text{and}\qquad\langle\phi_i,\phi_j\rangle=\delta_{ij},$$

where $\delta_{ij}=1$ if $i=j$ and zero, otherwise.

If (H1) is satisfied, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$ in H. It follows that the fractional powers A^α of A for $0 \leq \alpha \leq 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf., [22, pp. 69–75]). Hence, for convenience, we suppose that

$$\left\|e^{-tA}\right\| \leq M, \qquad \text{for all } t \geq 0,$$

and $0 \in \rho(-A)$, the resolvent set of $-A$.

The space $D(A^\alpha)$ which is denoted by H_α, is a Banach space endowed with the norm,

$$\|x\|_\alpha = \|A^\alpha x\|, \qquad x \in D(A^\alpha). \tag{2.5}$$

For $t \in [0,T]$, we denote by $C_t^\alpha := C([-\tau,t]; D(A^\alpha))$ endowed with the norm,

$$\|\psi\|_{t,\alpha} := \sup_{-\tau \leq \nu \leq t} \|\psi(\nu)\|_\alpha .$$

(H2) The function $\chi(t) \in D(A^\alpha)$, for all $t \in [-\tau,0]$ and χ is locally Hölder continuous on $[-\tau,0]$. We define

$$\tilde{\chi}(t) = \begin{cases} \chi(t), & t \in [-\tau,0], \\ \chi(0), & t \in [0,T_0]. \end{cases}$$

(H3) The nonlinear map $f : [0,T] \times H_\alpha \times C_0^\alpha \to X$ satisfies a local Lipschitz-like condition,

$$\left\|f(t,x,\psi) - f\left(t,y,\tilde{\psi}\right)\right\| \leq f_r(t)\left[\|x-y\|_\alpha + \left\|\psi - \tilde{\psi}\right\|_{0,\alpha}\right]$$

and

$$\|f(t,x,\psi)\| \leq f_r(t),$$

for all $t,s \in [0,T]$ and $x,y \in B_r(H_\alpha, \tilde{\chi}(\theta))$ where $-\tau \leq \theta \leq T$, $\psi, \tilde{\psi} \in B_r(C_0^\alpha, \tilde{\chi})$, where $f_r : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function depending on $r > 0$ and for z_0 in a Banach space $(Z, \|.\|_Z)$ and $r > 0$,

$$B_r(Z, z_0) = \{z \in Z : \|z - z_0\|_Z \leq r\}.$$

(H4) There exist positive constants $0 < \alpha < \beta < 1$ and $r > 0$, such that the function $A^\beta g$ is continuous for $(t,u,v) \in [0,T_0] \times H_\alpha \times C_0^\alpha$, such that

$$\left\|A^\beta g(t,u_1,u_2) - A^\beta g(s,v_1,v_2)\right\| \leq L\left\{|t-s|^\gamma + \|u_1 - v_1\|_\alpha + \|u_2 - v_2\|_{0,\alpha}\right\}$$

and

$$4L\left\|A^{\alpha-\beta}\right\| < 1,$$

for all $t,s \in [0,T]$, $0 < \gamma \leq 1$, and $x,y \in B_r(H_\alpha, \tilde{\chi}(\theta))$, where $-\tau \leq \theta \leq T$, $\psi, \tilde{\psi} \in B_r(C_0^\alpha, \tilde{\chi})$, L is a constant and for z_0 in a Banach space $(Z, \|.\|_Z)$ and $r > 0$,

$$B_r(Z, z_0) = \{z \in Z : \|z - z_0\|_Z \leq r\}.$$

3. APPROXIMATE SOLUTIONS AND CONVERGENCE

Let H_n denote the finite-dimensional subspace of H spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$ and let $P^n : H \longrightarrow H_n$ be the corresponding projection operator for $n = 0, 1, 2, \ldots$. Let $0 < T_0 \leq T$, α, β. and γ be such that

$$4L \left\|A^{\alpha-\beta}\right\| = \eta < 1, \tag{3.6}$$

$$B_1 = \max_{0 \leq t \leq T_0} \left\|A^{\beta} g\left(t, \bar{\chi}(0), \bar{\chi}_0\right)\right\|, \tag{3.7}$$

$$R_1 = 2R + 4\|\bar{\chi}\|_{T_0,\alpha}, \tag{3.8}$$

$$\left\|\left(e^{-tA} - I\right) A^{\alpha}\left(\chi(0) + g_n\left(0, u(0), u_0\right)\right)\right\| \leq \frac{R}{4}, \tag{3.9}$$

and

$$\left\|A^{\alpha-\beta}\right\| L\left\{T_0^{\gamma} + R_1\right\} \leq \frac{R}{4}. \tag{3.10}$$

For $t > 0$, $\|A^{\alpha} e^{-tA}\| \leq C_{\alpha} t^{-\alpha}$, where C_{α} is a positive constant. We have

$$\left\|\left(e^{-\xi A} - I\right) x\right\| \leq C'_{\nu} \xi^{\nu} \|x\|_{\nu}, \tag{3.11}$$

from Part (d) of Theorem 2.6.13 in [22] where $0 < \nu \leq 1$ and $x \in D(A^{\alpha})$. Let ν, where $0 < \nu < \min\{1 - \alpha, \beta - \alpha\}$, be a real number, then $A^{\alpha} y \in D(A^{\nu})$, for any $y \in D(A^{\alpha+\nu})$. For all $t, s \in (0, T_0]$, $t \geq s$, and $0 < \delta < 1$, we get the following inequalities,

$$\left\|\left(e^{-\delta A} - I\right) A^{\alpha} e^{-(t-s)A}\right\| \leq \frac{\bar{C}\delta^{\nu}}{(t-s)^{\alpha+\nu}}, \tag{3.12}$$

$$\left\|\left(e^{-\delta A} - I\right) A^{\alpha} e^{-tA}\right\| \leq C'_{\nu} \delta^{\nu} \left\|A^{\alpha+\nu} e^{-tA}\right\| \leq \frac{\bar{C}\delta^{\nu}}{t^{\alpha+\nu}}, \tag{3.13}$$

and

$$\left\|\left(e^{-\delta A} - I\right) A^{1+\alpha-\beta} e^{-(t-s)A}\right\| \leq \frac{\bar{C}\delta^{\nu}}{(t-s)^{1+\alpha+\nu-\beta}}, \tag{3.14}$$

where $\bar{C} = C'_{\nu} \max\{C_{\alpha+\nu}, C_{1+\alpha+\nu-\beta}\}$.

We choose T_0, α and β in such a way that

$$2C_{1+\alpha-\beta} L \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + 2C_{\alpha} f_R\left(T_0\right) \frac{T_0^{1-\alpha}}{1 - \alpha} < 1 - \eta \tag{3.15}$$

and

$$T_0 < \min\left\{d_1, d_2\right\}, \tag{3.16}$$

where

$$d_1 = \left\{\frac{R}{4}(\beta - \alpha)\left(C_{1+\alpha-\beta}\left(LR_1 + B_1\right)\right)^{-1}\right\}^{1/\beta-\alpha}, \tag{3.17}$$

$$d_2 = \left\{\frac{R}{4}(1 - \alpha)\left(f_R\left(T_0\right) C_{\alpha}\right)^{-1}\right\}^{1/1-\alpha}. \tag{3.18}$$

We define

$$g_n : [0, T] \times H_{\alpha} \times C_0^{\alpha} \longrightarrow H,$$

such that

$$g_n(t,u(t),u_t) = g(t,p^n u(t),P^n u_t)$$

and

$$f_n : [0,T] \times H_\alpha \times \mathcal{C}_0^\alpha \longrightarrow H,$$

given by

$$f_n(t,u(t),u_t) = f(t,p^n u(t),P^n u_t).$$

Let $A^\alpha : \mathcal{C}_t^\alpha \to \mathcal{C}_t$ be given by $(A^\alpha\psi)(t) = A^\alpha(\psi(t))$ and $(P^n u_t)(s) = P^n(u(t+s))$, for $s \in [-\tau,0]$, $t \in [0,T]$. We define a map F_n on $B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$ as follows,

$$(F_n u)(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau,0], \\ e^{-tA}(\bar{\chi}(0) + g_n(0,\bar{\chi}(0),\bar{\chi}_0)) - g_n(t,u(t),u_t) & \\ + \int_0^t Ae^{-(t-s)A} g_n(s,u(s),u_s)\,ds & \\ + \int_0^t e^{-(t-s)A} f_n(s,u(s),u_s)\,ds, & t \in [0,T_0], \end{cases} \tag{3.19}$$

for $u \in B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$.

THEOREM 3.1. *Let us assume that Assumptions (H1)–(H4) holds and $\chi(t) \in D(A)$, for all $t \in [-\tau,0]$. Then, there exists a unique $u_n \in B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$, such that $F_n u_n = u_n$, for each $n = 0,1,2,3,\ldots$, i.e., u_n satisfies the approximate integral equation,*

$$u_n(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau,0], \\ e^{-tA}(\bar{\chi}(0) + g_n(0,\bar{\chi}(0),(\bar{\chi})_0)) - g_n(t,u_n(t),(u_n)_t) & \\ + \int_0^t Ae^{-(t-s)A} g_n(s,u_n(s),(u_n)_s)\,ds & \\ + \int_0^t e^{-(t-s)A} f_n(s,u_n(s),(u_n)_s)\,ds, & t \in [0,T_0]. \end{cases} \tag{3.20}$$

PROOF. In order to proved the above theorem, first we need to show that the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau,T_0]$ into $D(A^\alpha)$ with respect to $\|.\|_\alpha$ norm and $F_n u \in B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$ for any $u \in B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$. Thus, for any $u \in B_R(\mathcal{C}_{T_0}^\alpha,\bar{\chi})$, and $t_1,t_2 \in [-\tau,0]$, we have

$$(F_n u)(t_1) - (F_n u)(t_2) = \bar{\chi}(t_1) - \bar{\chi}(t_2). \tag{3.21}$$

Now for $t_1,t_2 \in (0,T_0]$ with $t_1 < t_2$, we have

$$\begin{aligned} &\|(F_n u)(t_2) - (F_n u)(t_1)\|_\alpha \\ &\le \left\| \left(e^{-(t_2-t_1)A} - I\right) A^\alpha e^{-t_1 A} \right\| (\|\bar{\chi}(0)\| + \|g(0,P^n\bar{\chi}(0),P^n\bar{\chi}_0)\|) \\ &\quad + \|A^{\alpha-\beta}\| \, \|A^\beta g_n(t_2,u(t_2),u_{t_2}) - A^\beta g_n(t_1,u(t_1),u_{t_1})\| \\ &\quad + \int_0^{t_1} \left\| \left(e^{-(t_2-s)A} - e^{-(t_1-s)A}\right) A^{1+\alpha-\beta} \right\| \, \|A^\beta g_n(s,u(s),u_s)\|\,ds \\ &\quad + \int_{t_1}^{t_2} \left\| \left(e^{-(t_2-s)A}\right) A^{1+\alpha-\beta} \right\| \, \|A^\beta g_n(s,u(s),u_s)\|\,ds. \\ &\quad + \int_0^{t_1} \left\| \left(e^{-(t_2-s)A} - e^{-(t_1-s)A}\right) A^\alpha \right\| \, \|f_n(s,u(s),u_s)\|\,ds \\ &\quad + \int_{t_1}^{t_2} \left\| \left(e^{-(t_2-s)A}\right) A^\alpha \right\| \, \|f_n(s,u(s),u_s)\|\,ds. \end{aligned} \tag{3.22}$$

Now, we use inequality (3.12) to get the inequality given below,

$$\begin{aligned} &\int_0^{t_1} \left\| \left(e^{-(t_2-s)A} - e^{-(t_1-s)A}\right) A^\alpha \right\| \, \|f_n(s,u(s),u_s)\|\,ds \\ &\le \int_0^{t_1} \left\| \left(e^{-(t_2-t_1)A} - I\right) e^{-(t_1-s)A} A^\alpha \right\| \, \|f_n(s,u(s),u_s)\|\,ds \\ &\le \frac{\bar{C}(t_2-t_1)^\nu f_R(T_0) {T_0}^{1-(\alpha+\nu)}}{1-(\alpha+\nu)}. \end{aligned} \tag{3.23}$$

Similarly, we get inequality given below,

$$\int_{t_1}^{t_2} \left\| e^{-(t_2-s)A} A^{\alpha} \right\| \|f_n(s, u(s), u_s)\| \, ds \le \frac{C_{\alpha}(t_2-t_1)^{1-\alpha} f_R(T_0)}{(1-\alpha)}. \tag{3.24}$$

Also, we use the inequality (3.14) to get the inequality given below,

$$\begin{aligned} &\int_0^{t_1} \left\| \left(e^{-(t_2-s)A} - e^{-(t_1-s)A} \right) A^{1+\alpha-\beta} \right\| \left\| A^{\beta} g_n(s, u(s), u_s) \right\| ds \\ &\qquad \le \frac{\bar{C}(t_2-t_1)^{\nu} (LR_1 + B_1) T_0^{\beta-(\alpha+\nu)}}{\beta-(\alpha+\nu)}, \end{aligned} \tag{3.25}$$

where R_1 and B_1 are given by equalities (3.7) and (3.8). Also, we have

$$\int_{t_1}^{t_2} \left\| \left(e^{-(t_2-s)A} \right) A^{1+\alpha-\beta} \right\| \left\| A^{\beta} g_n(s, u(s), u_s) \right\| ds \le \frac{C_{1+\alpha-\beta}(t_2-t_1)^{\beta-\alpha} (LR_1 + B_1)}{\beta-\alpha}, \tag{3.26}$$

where R_1 and B_1 are same as given in the above inequality. Hence, from (3.21)–(3.26), the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau, T_0]$ into $D(A^{\alpha})$ with respect to $\|\cdot\|_{\alpha}$ norm.

Now, for $t \in [-\tau, 0]$,

$$(F_n u)(t) - \bar{\chi}(t) = 0.$$

Now, for $t \in (0, T_0]$, we have

$$\begin{aligned} &\|(F_n u)(t) - \bar{\chi}(t)\|_{\alpha} \\ &\quad \le \left\| \left(e^{-tA} - I\right) A^{\alpha} \left(\bar{\chi}(0) + g_n(0, \bar{\chi}(0), \bar{\chi}_0)\right) \right\| \\ &\qquad + \left\| A^{\alpha-\beta} \right\| \left\| A^{\beta} g_n(0, \bar{\chi}(0), \bar{\chi}_0) - A^{\beta} g_n(t, u(t), u_t) \right\| \\ &\qquad + \int_0^t \left\| A^{1+\alpha-\beta} e^{-(t-s)A} \right\| \left\| A^{\beta} g_n(s, u(s), u_s) \right\| ds \\ &\qquad + \int_0^t \left\| e^{-(t-s)A} A^{\alpha} \right\| [\|f_n(s, u(s), u_s)\| \, ds \\ &\quad \le \left\| \left(e^{-tA} - I\right) A^{\alpha} \left(\bar{\chi}(0) + g_n(0, \bar{\chi}(0), \bar{\chi}_0)\right) \right\| \\ &\qquad + \left\| A^{\alpha-\beta} \right\| L \left\{ T_0^{\gamma} + \|u(t) - \bar{\chi}(0)\|_{\alpha} + \|u_t - \bar{\chi}_0\|_{0,\alpha} \right\} \\ &\qquad + C_{1+\alpha-\beta}(LR_1 + B_1) \frac{T_0^{\beta-\alpha}}{\beta-\alpha} + C_{\alpha} f_R(T_0) \frac{T_0^{1-\alpha}}{1-\alpha} \\ &\quad \le R. \end{aligned}$$

Hence,

$$\|F_n u - \bar{\chi}\|_{T_0,\alpha} \le R.$$

Thus,

$$F_n : B_R\left(C_{T_0}^{\alpha}, \bar{\chi}\right) \to B_R\left(C_{T_0}^{\alpha}, \bar{\chi}\right).$$

Now, for any $u, v \in B_R(C_{T_0}^{\alpha}, \bar{\chi})$ and $t \in [-\tau, 0]$, we have

$$\|F_n u(t) - F_n v(t)\|_{\alpha} = 0. \tag{3.27}$$

Now, if $t \in (0, T_0]$ and $u, v \in B_R(C_{T_0}^{\alpha}, \bar{\chi})$, then

$$\begin{aligned} \|(F_n u)(t) - (F_n v)(t)\|_{\alpha} &\le \left\| A^{\alpha-\beta} \right\| \left\| A^{\beta} g_n(t, u(t), u_t) - A^{\beta} g_n(t, v(t), v_t) \right\|_{\alpha} \\ &+ \int_0^t \left\| e^{-(t-s)A} A^{1+\alpha-\beta} \right\| \left\| A^{\beta} g_n(s, u(s), u_s) - A^{\beta} g_n(s, v(s), v_s) \right\| ds \\ &+ \int_0^t \left\| e^{-(t-s)A} A^{\alpha} \right\| \|f_n(s, u(s), u_s) - f_n(s, v(s), v_s)\| \, ds. \end{aligned} \tag{3.28}$$

Here, we have

$$\left\|A^{\beta}g_n\left(t,u\left(t\right),u_t\right)-A^{\beta}g_n\left(t,v\left(t\right),v_t\right)\right\|\le 2L\left\|u-v\right\|_{T_0,\alpha}, \tag{3.29}$$

$$\left\|f_n\left(s,u\left(s\right),u_s\right)-f_n\left(s,v\left(s\right),v_s\right)\right\|\le 2f_R\left(T_0\right)\left\|u-v\right\|_{T_0,\alpha}. \tag{3.30}$$

Thus, from inequalities (3.29) and (3.30), the inequality (3.28) becomes

$$\begin{aligned}&\left\|F_nu\left(t\right)-F_nv\left(t\right)\right\|_{\alpha}\\ &\le\left(\left\|A^{\alpha-\beta}\right\|2L+2C_{1+\alpha-\beta}L\frac{T_0^{\beta-\alpha}}{\beta-\alpha}+2C_{\alpha}f_R\left(T_0\right)\frac{T_0^{1-\alpha}}{1-\alpha}\right)\left\|u-v\right\|_{T_0,\alpha}.\end{aligned} \tag{3.31}$$

Thus, for all $t\in[-\tau,T_0]$, we have

$$\begin{aligned}&\sup_{-\tau\le t\le T_0}\left\|F_nu\left(t\right)-F_nv\left(t\right)\right\|_{\alpha}\\ &\le\left(\left\|A^{\alpha-\beta}\right\|2L+2C_{1+\alpha-\beta}L\frac{T_0^{\beta-\alpha}}{\beta-\alpha}+2C_{\alpha}\bar{F}_R\left(T_0\right)\frac{T_0^{1-\alpha}}{1-\alpha}\right)\left\|u-v\right\|_{T_0,\alpha}.\end{aligned} \tag{3.32}$$

Thus, we get

$$\left\|F_nu-F_nv\right\|_{T_0\alpha}<\left\|u-v\right\|_{T_0,\alpha}.$$

Hence, there exist a unique $u_n\in B_R(\mathcal{C}^{\alpha}_{T_0},\tilde{\chi})$, such that $F_nu_n=u_n$, which satisfies the approximate integral equation (3.20), hence, the theorem is proved.

COROLLARY 1. *If Assumptions (H1)–(H4) hold and $\chi(t)\in D(A)$, for all $t\in[-\tau,0]$ then $u_n(t)\in D(A^{\nu})$, for all $t\in[-\tau,T_0]$ where $0\le\nu\le\beta<1$.*

PROOF. For all $t\in[-\tau,0]$, it is obvious. Therefore, we left with $t\in(0,T]$. From Theorem 3.1, we have the existence of a unique $u_n\in B_R(\mathcal{C}^{\alpha}_{T},\tilde{x})$ satisfying (3.20). Part (a) of Theorem 2.6.13 in [22] implies that, for $t>0$ and $0\le\vartheta<1$, $e^{-tA}:H\to D(A^{\vartheta})$ and for $0\le\vartheta\le\beta<1$, $D(A^{\beta})\subseteq D(A^{\vartheta})$. (H4) implies that the map $t\mapsto A^{\beta}g(t,u_n(t),u_t)$ is Hölder continuous on $[0,T]$ with the exponent $\rho=\min\{\gamma,\vartheta\}$ since the Höder continuity of u_n can be easily established using the similar argument from (3.22) to (3.26). It follows that (cf., Theorem 4.3.2 in [22])

$$\int_0^t e^{-(t-s)A}A^{\beta}g_n\left(s,u_n\left(s\right),u_{ns}\right)ds\in D\left(A\right).$$

Also, from Theorem 1.2.4 in [22], we have $e^{-tA}x\in D(A)$ if $x\in D(A)$. The required result follows from these facts and the fact that $D(A)\subseteq D(A^{\vartheta})$, for $0\le\vartheta\le 1$.

COROLLARY 2. *If Assumptions (H1)–(H4) hold and $\chi(t)\in D(A)$, for all $t\in[-\tau,0]$, then there exist a constant M_0, independent of n, such that*

$$\left\|A^{\nu}u_n\left(t\right)\right\|\le M_0,$$

for all $-\tau\le t\le T_0$ and $0<\nu<\beta<1$.

PROOF. Now, applying A^{ν} on both the sides in (3.20), and for any $t\in[-\tau,o]$, we have

$$\left\|u_n\left(t\right)\right\|_{\nu}\le\left\|\bar{\chi}\left(t\right)\right\|_{\nu},\qquad\text{for all } t\in\left[-\tau,0\right]. \tag{3.33}$$

Now, for $t\in(0,T_0]$, we have

$$\begin{aligned}\left\|u_n\left(t\right)\right\|_{\nu}&\le M\left(\left\|\chi\left(0\right)\right\|_{\nu}+\left\|g_n\left(0,u_n\left(0\right),\left(u_n\right)_0\right)\right\|_{\nu}\right)\\ &\quad+\left\|A^{\nu-\beta}\right\|\left(LR_1+B_1\right)+C_{1+\nu-\beta}\left(LR_1+B_1\right)\frac{T_0^{\beta-\nu}}{\beta-\nu}+C_{\nu}f_R\left(T_0\right)\frac{T_0^{1-\nu}}{1-\beta}\\ &=M_0'.\end{aligned} \tag{3.34}$$

Thus, we get the required result.

COROLLARY 3.4. *If Assumptions (H1)–(H4) hold and $\chi(t) \in D(A)$, for all $t \in [-\tau, 0]$ then the sequence $\{u_n\} \subset \mathcal{C}^{\alpha}_{T_0}$ is a Cauchy sequence and therefore converges to a unique function $u \in \mathcal{C}^{\alpha}_{T_0}$.*

PROOF. Let $0 < \alpha < \nu < \beta$. For $n \geq m$ and $t \in (0, T_0]$, we have

$$\begin{aligned}
&\left\|f_n\left(t, u_n(t), (u_n)_t\right) - f_m\left(t, u_m(t), (u_m)_t\right)\right\| \\
&\quad \leq \left\|f_n\left(t, u_n(t), (u_n)_t\right) - f_n\left(t, u_m(t), (u_m)_t\right\| \\
&\qquad + \left\|f_n\left(t, u_m(t), (u_m)_t\right) - f_m\left(t, u_m(t), (u_m)_t\right\| \\
&\quad \leq F_R(T_0)\left[\left\|u_n(t) - u_m(t)\right\|_\alpha + \left\|(u_n)_t - (u_m)_t\right\|_{0,\alpha} + \frac{1}{\lambda_m^{\nu-\alpha}}\left\|A^\nu u_m(t)\right\|\right. \\
&\qquad \left. + \frac{1}{\lambda_m^{\nu-\alpha}}\left\|(u_m)_t\right\|_{0,\nu}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left\|A^\beta g_n\left(t, u_n(t), (u_n)_t\right) - A^\beta g_m\left(t, u_m(t), (u_m)_t\right)\right\| \\
&\quad \leq \left\|A^\beta g_n\left(t, u_n(t), (u_n)_t\right) - A^\beta g_n\left(t, u_m(t), (u_m)_t\right)\right\| \\
&\qquad + \left\|A^\beta g_n\left(t, u_m(t), (u_m)_t\right) - A^\beta g_m\left(t, u_m(t), (u_m)_t\right)\right\| \\
&\quad \leq L\left[\left\|u_n(t) - u_m(t)\right\|_\alpha + \left\|(u_n)_t - (u_m)_t\right\|_{0,\alpha} + \frac{1}{\lambda_m^{\nu-\alpha}}\left\|A^\nu u_m(t)\right\|\right. \\
&\qquad \left. + \frac{1}{\lambda_m^{\nu-\alpha}}\left\|(u_m)_t\right\|_{0,\nu}\right].
\end{aligned} \tag{3.35}$$

Now for $t \in [-\tau, 0]$ we have

$$u_n(t) - u_m(t) = 0.$$

Now, for $0 < t_0' < t < T_0$, we may write

$$\begin{aligned}
\left\|u_n(t) - u_m(t)\right\|_\alpha &\leq \left\|e^{-tA}A^\alpha\left(g_n\left(0, \bar\chi(0), (\bar\chi)_0\right) - g_m\left(0, \bar\chi(0), \bar\chi_0\right)\right)\right\| \\
&\quad + \left\|A^{\alpha-\beta}\right\|\left\|A^\beta g_n\left(t, u_n(t), (u_n)_t\right) - A^\beta g_m\left(t, u_m(t), (u_m)_t\right)\right\| \\
+ \left(\int_0^{t_0'} + \int_{t_0'}^t\right)&\left\|A^{1+\alpha-\beta}e^{-(t-s)A}\right\|\left\|A^\beta g_n\left(t, u_n(t), (u_n)_t\right) - A^\beta g_m\left(t, u_m(t), (u_m)_t\right)\right\| ds \\
&+ \left(\int_0^{t_0'} + \int_{t_0'}^t\right)\left\|A^\alpha e^{-(t-s)A}\right\|\left\|f_n\left(t, u_n(t), (u_n)_t\right) - f_m\left(t, u_m(t), (u_m)_t\right)\right\| ds.
\end{aligned} \tag{3.36}$$

We estimate the first term as

$$\begin{aligned}
&\left\|e^{-tA}A^\alpha\left(g_n\left(0, \bar\chi(0), \bar\chi_0\right) - g_m\left(0, \bar\chi(0), \bar\chi_0\right)\right)\right\| \\
&\quad \leq M\left\|A^{\alpha-\beta}\right\|\left\|A^\beta g_n\left(0, P^n\bar\chi(0), P^n\bar\chi_0\right) - A^\beta g\left(0, P^m\bar\chi(0), P^m\bar\chi_0\right)\right\| \\
&\quad \leq M\left\|A^{\alpha-\beta}\right\| L\left[\left\|(P^n - P^m)A^\alpha\bar\chi(0)\right\| + \left\|(P^n - P^m)\bar\chi_0\right\|_{0,\alpha}\right].
\end{aligned}$$

The estimation of the second term is same as the estimation of inequality (3.35), and hence,

$$\begin{aligned}
&\left\|A^{\alpha-\beta}\right\|\left\|A^\beta g_n\left(t, u_n(t), (u_n)_t\right) - A^\beta g_m\left(t, u_m(t), (u_m)_t\right)\right\| \\
&\qquad \leq 2L\left\|A^{\alpha-\beta}\right\|\left[\left\|u_n - u_m\right\|_{T_0,\alpha} + \frac{1}{\lambda_m^{\nu-\alpha}}M_0\right].
\end{aligned}$$

The first integral is given by

$$\begin{aligned}
&\int_0^{t_0'}\left\|A^{1+\alpha-\beta}e^{-(t-s)A}\right\|\left\|A^\beta g_n\left(s, u_n(s), (u_n)_s\right) - A^\beta g_m\left(s, u_m(s), (u_m)_s\right)\right\| ds \\
&\qquad \leq 2C_{1+\alpha-\beta}\left(LR_1 + B_1\right)\left(t_0 - t_0'\right)^{-(1+\alpha-\beta)} t_0'
\end{aligned}$$

and the third integral is given by

$$\int_0^{t_0'} \left\|A^{\alpha} e^{-(t-s)A}\right\| \|f_n(s, u_n(s), (u_n)_s) - f_m(s, u_m(s), (u_m)_s)\| \, ds$$
$$\leq 2C_\alpha f_R(T_0)(t_0 - t_0')^{-\alpha} t_0'.$$

Second term is calculated as follows

$$\begin{aligned}
&\int_{t_0'}^{t} \left\|A^{1+\alpha-\beta} e^{-(t-s)A}\right\| \left\|A^{\beta} g_n(s, u_n(s), (u_n)_s) - A^{\beta} g_m(s, u_m(s), (u_m)_s\right\| \, ds \\
&\leq C_{1+\alpha-\beta} L \int_{t_0'}^{t} (t-s)^{-(1+\alpha-\beta)} \Bigg[\|u_n(s) - u_m(s)\|_\alpha \\
&\quad + \|(u_n)_s - (u_m)_s\|_{0,\alpha} + \frac{1}{\lambda_m^{\nu-\alpha}} \|A^{\nu} u_m(s)\| + \frac{1}{\lambda_m^{\nu-\alpha}} \|u_{ms}\|_{0,\nu} \Bigg] \, ds \\
&\leq C_{1+\alpha-\beta} 2L \left\{ \frac{M_0 T_0^{\beta-\alpha}}{\lambda_m^{\nu-\alpha}(\beta-\alpha)} + \int_{t_0'}^{t} (t-s)^{-(1+\alpha-\beta)} \|u_n - u_m\|_{s,\alpha} \, ds \right\}.
\end{aligned}$$

Fourth term calculated as follows,

$$\int_{t_0'}^{t} \left\|A^{\alpha} e^{-(t-s)A}\right\| \|f_n(s, u_n(s), (u_n)_s) - f_m(s, u_m(s), (u_m)_s)\| \, ds$$
$$\leq 2C_\alpha f_R(T_0) \left\{ \frac{M_0 T_0^{1-\alpha}}{\lambda_m^{\nu-\alpha}(1-\alpha)} + \int_{t_0'}^{t} (t-s)^{-\alpha} \|u_n - u_m\|_{s,\alpha} \, ds \right\}.$$

Therefore,

$$\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha &\leq M \left\|A^{\alpha-\beta}\right\| L \left[\|(P^n - P^m) A^{\alpha} \bar{\chi}(0)\| + \|(P^n - P^m) \bar{\chi}_0\|_{0,\alpha} \right] \\
&\quad + 2L \left\|A^{\alpha-\beta}\right\| \left[\|u_n - u_m\|_{T_0,\alpha} + \frac{1}{\lambda_m^{\nu-\alpha}} M_0 \right] \\
&\quad + 2C_{1+\alpha-\beta}(LR_1 + B_1)(t_0 - t_0')^{-(1+\alpha-\beta)} t_0', + 2C_\alpha f_R(T_0)(t_0 - t_0')^{-\alpha} t_0' \\
&\quad + C_{1+\alpha-\beta} 2L \left\{ \frac{M_0 T_0^{\beta-\alpha}}{\lambda_m^{\nu-\alpha}(\beta-\alpha)} + \int_{t_0'}^{t} (t-s)^{-(1+\alpha-\beta)} \|u_n - u_m\|_{s,\alpha} \, ds \right\} \\
&\quad + 2C_\alpha f_R(T_0) \left\{ \frac{M_0 T_0^{1-\alpha}}{\lambda_m^{\nu-\alpha}(1-\alpha)} + \int_{t_0'}^{t} (t-s)^{-\alpha} \|u_n - u_m\|_{s,\alpha} \, ds \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha &\leq A_1 + 2L \left\|A^{\alpha-\beta}\right\| \|u_n - u_m\|_{T_0,\alpha} \\
&\quad + A_2 t_0' + \frac{A_3}{\lambda_m^{\nu-\alpha}} + A_4 \int_{t_0'}^{t} (t-s)^{-\alpha} \|u_n - u_m\|_{s,\alpha} \, ds
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
A_1 &= M \left\|A^{\alpha-\beta}\right\| \left[\|(p^n - p^m) \bar{\chi}(0)\|_\alpha + \|(p^n - p^m) \bar{\chi}_0\|_{0,\alpha} \right], \\
A_2 &= 2C_{1+\alpha-\beta}(LR_1 + B_1)(t_0 - t_0')^{-(1+\alpha-\beta)} + 2C_\alpha f_R(T_0)(t_0 - t_0')^{-\alpha}, \\
A_3 &= 2L \left\|A^{\alpha-\beta}\right\| M_0 + 2LC_{1+\alpha-\beta} T_0^{\beta-\alpha} \frac{M_0}{(\beta-\alpha)} + 2C_\alpha f_R(T_0) M_0 \frac{T_0^{1-\alpha}}{(1-\alpha)},
\end{aligned}$$

and

$$A_4 = 2LC_{1+\alpha-\beta} + 2C_\alpha f_R(T_0).$$

Now, we replace t by $t+\theta$ in inequality (3.37) where $\theta \in [t_0' - t, 0]$, we get

$$\begin{aligned} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha &\le A_1 + 2L\left\|A^{\alpha-\beta}\right\| \|u_n - u_m\|_{T_0,\alpha} + A_2 t_0' \\ &+ \frac{A_3}{\lambda_m^{\nu-\alpha}} + A_4 \int_{t_0'}^{t+\theta} (t+\theta-s)^{-\alpha} \|u_n - u_m\|_{s,\alpha}\, ds. \end{aligned} \tag{3.38}$$

We put $s - \theta = \gamma$ in inequality (3.38) and we get

$$\begin{aligned} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha &\le A_1 + 2L\left\|A^{\alpha-\beta}\right\| \|u_n - u_m\|_{T_0,\alpha} \\ &\quad + A_2 t_0' + \frac{A_3}{\lambda_m^{\nu-\alpha}} + A_4 \int_{t_0'-\theta}^{t} (t-\gamma)^{-\alpha} \|u_n - u_m\|_{\gamma,\alpha}\, d\nu \\ &\le A_1 + 2L\left\|A^{\alpha-\beta}\right\| \|u_n - u_m\|_{T_0,\alpha} \\ &\quad + A_2 t_0' + \frac{A_3}{\lambda_m^{\nu-\alpha}} + A_4 \int_{t_0'}^{t} (t-\gamma)^{-\alpha} \|u_n - u_m\|_{\gamma,\alpha}\, d\nu. \end{aligned} \tag{3.39}$$

Thus,

$$\begin{aligned} \sup_{t_0'-t\le\theta\le 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha &\le A_1 + 2L\left\|A^{\alpha-\beta}\right\| \|u_n - u_m\|_{T_0,\alpha} \\ &+ A_2 t_0' + \frac{A_3}{\lambda_m^{\nu-\alpha}} + A_4 \int_{t_0'}^{t} (t-\gamma)^{-\alpha} \|u_n - u_m\|_{\gamma,\alpha}\, d\nu. \end{aligned} \tag{3.40}$$

Now, we have

$$\begin{aligned} \sup_{-\tau-t\le\theta\le 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha &\le \sup_{0\le\theta+t\le t_0'} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\ &+ \sup_{t_0'-t\le\theta\le 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha . \end{aligned} \tag{3.41}$$

Using inequalities (3.40) and (3.36) in the above inequality, we get

$$\begin{aligned} \sup_{-\tau\le t+\theta\le t} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha &\le \frac{1}{(1-4L\left\|A^{\alpha-\beta}\right\|)} 2A_1 + \frac{1}{(1-4L\left\|A^{\alpha-\beta}\right\|)} 2A_2 t_0' \\ + \frac{1}{(1-4L\left\|A^{\alpha-\beta}\right\|)} \frac{(A_3+A_5)}{\lambda_m^{\nu-\alpha}} &+ \frac{1}{(1-4L\left\|A^{\alpha-\beta}\right\|)} A_4 \int_{t_0'}^{t} (t-\gamma)^{-\alpha} \|u_n - u_m\|_{\gamma,\alpha}\, d\nu, \end{aligned} \tag{3.42}$$

where $A_5 = 2L\|A^{\alpha-\beta}\|M_0$.

Application of Gronwall's inequality to the above inequality, letting $m, n \to \infty$ and as t_0 is arbitrary small gives the required result. This completes the proof of the theorem.

With the help of Theorems 3.1 and 3.4, we may state the following existence, uniqueness, and convergence result.

THEOREM 3.5. *Suppose that (H1)–(H4) are satisfying and $\chi(t) \in D(A)$ for all $t \in [-\tau, 0]$ then there exist functions $u_n \in C([-\tau, T_0]; H_n)$ and $u \in C([-\tau, T_0]; H)$ satisfying*

$$u_n(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}\left(\bar{\chi}(0) + g_n(0, \bar{\chi}(0), \bar{\chi}_0) - g_n(t, u_n(t), (u_n)_t)\right) & \\ + \int_0^t A e^{-(t-s)A} g_n(s, u_n(s), (u_n)_s)\, ds & \\ + \int_0^t e^{-(t-s)A} f_n(s, u_n(s), (u_n)_s)\, ds, & t \in [0, T_0], \end{cases} \tag{3.43}$$

and

$$u(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}\left(\bar{\chi}(0) + g(0, \bar{\chi}(0), \bar{\chi}_0)\right) - g_n(t, u(t), u_t) & \\ + \int_0^t A e^{-(t-s)A} g(s, u(s), u_s)\, ds & \\ + \int_0^t e^{-(t-s)A} f(s, u(s), u_s)\, ds, & t \in [0, T_0], \end{cases} \tag{3.44}$$

such that $u_n \to u$ in $C([-\tau, T_0]; H)$ as $n \to \infty$, where f_n and g_n are as defined earlier.

4. FAEDO-GALERKIN APPROXIMATIONS

We knows from the previous sections that for any $-\tau \le T_0 \le T$, we have a unique $u \in C^{\alpha}_{T_0}$ satisfying the integral equation,

$$u(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}\left(\bar{\chi} + g\left(0, \bar{\chi}(0), \bar{\chi}_0\right)\right) - g_n\left(t, u(t), u_t\right) \\ + \int_0^t Ae^{-(t-s)A} g\left(s, u(s), u_s\right) ds \\ + \int_0^t e^{-(t-s)A} f\left(s, u(s), u_s\right) ds, & t \in [0, T_0]. \end{cases} \tag{4.45}$$

Also, there is a unique solutions $u_n \in C^{\alpha}_{T_0}$ of the approximate integral equation,

$$u_n(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}\left(\bar{\chi}(0) + g_n\left(0, \bar{\chi}(0), \bar{\chi}_0\right)\right) - g_n\left(t, u_n(t), (u_n)_t\right) \\ + \int_0^t Ae^{-(t-s)A} g_n\left(s, u_n(s), (u_n)_s\right) ds \\ + \int_0^t e^{-(t-s)A} f_n\left(s, u_n(s), (u_n)_s\right) ds & t \in [0, T_0]. \end{cases} \tag{4.46}$$

Now, Faedo-Galerkin approximation is given by $\bar{u}_n = P^n u_n$ satisfying

$$\bar{u}_n(t) = \begin{cases} p^n\bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}p^n\left(\bar{\chi}(0) + g_n\left(0, \bar{\chi}(0), \bar{\chi}_0\right)\right) - g_n\left(t, u_n(t), (u_n)_t\right) \\ + \int_0^t Ae^{-(t-s)A} p^n g_n\left(s, u_n(s), (u_n)_s\right) ds \\ + \int_0^t e^{-(t-s)A} p^n f_n\left(s, u_n(s), (u_n)_s\right) ds, & t \in [0, T_0], \end{cases} \tag{4.47}$$

where f_n and g_n are as defined earlier.

If the solution $u(t)$ to (4.45) exits on $-\tau \le t \le T_0$, then it has the representation,

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)\,\phi_i, \tag{4.48}$$

where $\alpha_i(t) = \langle u(t), \phi_i\rangle$, for all $i = 0, 1, 2, 3, \dots$, and

$$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\,\phi_i, \tag{4.49}$$

where $\alpha_i^n(t) = \langle \bar{u}_n(t), \phi_i\rangle$, for all $i = 0, 1, 2, 3, \dots$.

As a consequence of Theorems 3.1 and 3.4, we have the following result.

THEOREM 4.1. *Suppose that (H1)–(H4) hold and $\chi(t) \in D(A)$, for all $t \in [-\tau, 0]$, then there exist functions $\bar{u}_n \in C([-\tau, T_0]; H_n)$ and $u \in C([-\tau, T_0]; H)$, such that*

$$\bar{u}_n(t) = \begin{cases} p^n\bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}p^n\left(\bar{\chi}(0) + g_n\left(0, \bar{\chi}(0), \bar{\chi}_0\right)\right) - g_n\left(t, u_n(t), (u_n)_t\right) \\ + \int_0^t Ae^{-(t-s)A} p^n g_n\left(s, u_n(s), (u_n)_s\right) ds \\ + \int_0^t e^{-(t-s)A} p^n f_n\left(s, u_n(s), (u_n)_s\right) ds, & t \in [0, T_0] \end{cases} \tag{4.50}$$

and

$$u(t) = \begin{cases} \bar{\chi}(t), & t \in [-\tau, 0], \\ e^{-tA}\left(\bar{\chi}(0) + g\left(0, \bar{\chi}(0), \bar{\chi}_0\right)\right) - g\left(t, u(t), u_t\right) \\ + \int_0^t Ae^{-(t-s)A} g\left(s, u(s), u_s\right) ds \\ + \int_0^t e^{-(t-s)A} f\left(s, u(s), u_s\right) ds, & t \in [0, T_0]. \end{cases} \tag{4.51}$$

Here, $\bar{u}_n \to u$ in $C([-\tau, T_0]; H)$ as $n \to \infty$, where f_n and g_n are as defined earlier.

THEOREM 4.2. *Let (H1)–(H4) hold, then we have the following. If $h(t) \in D(A)$, for all $t \in [-\tau, 0]$, then for any $-\tau \le T_0 \le T$, $\lim_{n\to\infty} \sup_{-\tau\le t\le T_0} [\sum_{i=0}^{n} \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2] = 0$.*

PROOF. We have

$$A^{\alpha}\left[u(t) - \bar{u}_n(t)\right] = A^{\alpha}\left[\sum_{i=0}^{\infty}\left\{\alpha_i(t) - \alpha_i^n(t)\right\}\phi_i\right] = \sum_{i=0}^{\infty}\lambda_i^{\alpha}\left\{\alpha_i(t) - \alpha_i^n(t)\right\}\phi_i. \tag{4.52}$$

Thus, we have

$$\left\|A^{\alpha}\left[u(t) - \bar{u}_n(t)\right]\right\|^2 \ge \sum_{i=0}^{n}\lambda_i^{2\alpha}\left|\alpha_i(t) - \alpha_i^n(t)\right|^2. \tag{4.53}$$

Hence, as a consequence of Theorem 4.1, we have the required result.

5. APPLICATIONS

Let $X = L^2(0,1)$ and $\tau > 0$. Consider the partial differential equation,

$$\begin{gathered}
\partial_t \Big(w(t,x) + \Big(f_1(t,x) \\
+ \left(\int_0^1 h_1(w(t,x), \partial_x w(t,x))\, dx\right) \int_{-\tau}^{0} k_1(-\theta)\, h_2(w(t+\theta,x), \partial_x w(t+\theta,x))\, d\theta\Big)\Big) \\
-\partial_x^2 w(t,x) = f_2(t,x) \\
+ \left(\int_0^1 h_3(w(t,x), \partial_x w(t,x))\, dx\right) \int_{-\tau}^{0} k_2(-\theta)\, h_4(w(t+\theta,x), \partial_x w(t+\theta,x))\, d\theta, \\
x \in (0,1), \qquad t > 0, \\
w(t,x) = \chi(t,x), \qquad t \in [-\tau, 0], \quad x \in (0,1), \\
w(t,0) = w(t,1) = 0, \qquad t \in [0,T], \quad 0 < T < \infty,
\end{gathered} \tag{5.54}$$

where h_1, h_2, h_3, h_4, f_1, and f_2 are real valued smooth functions, k_1 and k_2 are square integrable functions and χ is locally Hölder continuous function on $[-\tau, 0]$ satisfies the condition $\chi(0,0) = \chi(0,1) = 0$.

We define an operator A, as follows,

$$Au = -u'', \qquad \text{with } u \in D(A) = \left\{u \in H_0^1(0,1) \cap H^2(0,1) : u'' \in X\right\}. \tag{5.55}$$

Here, clearly, the operator A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now, we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm,

$$\|x\|_{1/2} := \left\|A^{1/2}x\right\|, \qquad x \in D\left(A^{1/2}\right),$$

and we denote this space by $X_{1/2}$. Also, for $t \in [0,T]$, we denote

$$C_t^{1/2} = C\left([-\tau, t]; D\left(A^{1/2}\right)\right),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} := \sup_{-\tau\le\eta\le t} \|\psi(\eta)\|_{1/2}, \qquad \psi \in C_t^{1/2}.$$

We observe some properties of the operators A and $A^{1/2}$ defined by (5.55) (cf., [22], for more details). For $u \in D(A)$ and $\lambda \in \mathbb{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\langle -u'', u\rangle = |u'|_{L^2}^2 = \lambda\, |u|_{L^2}^2,$$

so, $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form,

$$u(x) = C\cos\left(\sqrt{\lambda}x\right) + D\sin\left(\sqrt{\lambda}x\right),$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is given by

$$u_n(x) = D\sin\left(\sqrt{\lambda_n}x\right).$$

We have $\langle u_n, u_m\rangle = 0$, for $n \neq m$ and $\langle u_n, u_n\rangle = 1$, and hence, $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$, such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x), \qquad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty, \qquad \text{and} \qquad \sum_{n\in\mathbb{N}} (\lambda_n)^2 (\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\, \alpha_n\, u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$.

Equation (5.54) can be reformulated as the following abstract equation in $X = L^2(0,1)$:

$$\begin{aligned} \frac{d}{dt}\left(u(t) + g(t, u(t), u_t)\right) + Au(t) &= f(t, u(t), u_t) \qquad && t > 0, \\ u(t) &= \chi(t), && t \in [-\tau, 0], \end{aligned} \tag{5.56}$$

where $u(t) = w(t,.)$ that is $u(t)(x) = w(t,x)$, $u_t(\theta)(x) = w(t+\theta, x)$, $t \in [0,T]$, $\theta \in [-\tau, 0]$, $x \in (0,1)$ and $u(t,x) = \chi(t,x)$, $t \in [-\tau, 0]$, $x \in (0,1)$. Operator A is as define in equation (5.55) and the function $g : [0,T] \times X_{1/2} \times C_0^{1/2} \to X$, is given by

$$\begin{aligned} g(t, \psi, \xi)(x) &= f_1(t,x) \\ &+ \left(\int_0^1 \left(h_1\left(\psi(x),\ \psi'(x)\right) dx\right)\right) \int_{-\tau}^0 k(-\theta)\, h_2\left(\xi(\theta)(x),\ \partial_x\left(\xi(\theta)(x)\right)\right) d\theta. \end{aligned} \tag{5.57}$$

Also f_1, define from $[0,T] \times (0,1)$ into $\mathbb{R}$, is such that $f_1(0,.) \in L^2(0,1)$ and satisfies the following property,

$$|f_1(t,x) - f_1(s,x)| \leq k_3(x)|t-s|^\theta, \qquad \text{for all } t, s \in \mathbb{R}, \quad \text{a.e. } x \in (0,1),$$

where $k_3 \in L^2(0,1)$. We can easily verified that the function g is satisfied Assumption (H4). In the similar manner, we can define the function f which satisfies Assumption (H3).

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